Operation-Based Signal-Flow AC Analysis of Switching DC–DC Converters in CCM and DCM

Dongwon Kwon, Graduate Student Member, IEEE, and Gabriel A. Rincón-Mora, Senior Member, IEEE
Georgia Tech Analog, Power, and Energy IC Research
E-mail: dkwon@ece.gatech.edu, rincon-mora@ece.gatech.edu

Abstract—Ensuring stable operation is perhaps one of the most challenging aspects of designing a power-supply circuit because the load is dynamically unpredictable and widely variable. Unfortunately, the nonlinear dynamics of switch-mode converters accentuate these difficulties, especially when considering conventional analytical techniques rely on abstract mathematics to describe ac switching events that convey little operational insight into the circuit. This paper proposes an operation-based signal-flow ac analysis approach that is both sufficiently general to apply to most switched-inductor converters in continuous- and discontinuous-conduction modes (CCM and DCM) and insightful enough to derive directly from the waveforms of the circuit, not from the system of differential equations that govern them. To this end, after presenting the technique, the paper applies it to a relatively simple topology like the buck converter to prove its efficacy and then to a more complex counterpart like the non-inverting buck-boost converter to show the ease with which a more complicated system can be analyzed.

I. AC ANALYSIS OF SWITCHING CONVERTERS

Switching dc–dc converters employ negative feedback to regulate the output against variations in load power and (input) line voltage. As a result, understanding the ac characteristics of the system across all possible operating conditions is crucial to ensure the stability of the circuit when subjected to sudden load and/or line changes. Describing these conditional states with equations that reflect the actual operation of the circuit is important to fully comprehend and optimize design tradeoffs. Unfortunately, correlating (insightfully) the nonlinear dynamics of switch-mode circuits is difficult. Therefore, conventional schemes [1-13] first find the operating point through averaging techniques and subsequently linearize the system at that operating point to describe the small-signal operation with equations.

Although the resulting math presents an excellent vehicle to analyze and decipher the stability requirements of the system, the equations remain abstract (with respect to the operation of the circuit). The designer is therefore less able to counter the root-cause factors producing ill-fated effects in the system’s response. This paper proposes to bridge this gap by presenting a technique that allows the designer to extract operationally insightful signal-flow graphs (SFGs) directly from inductor-based switching converters operating in both continuous- and discontinuous-conduction modes (CCM and DCM). To this end, for review, Sections II and III of the paper overview the stability objectives and the conventional means of analyzing switching supplies employing negative feedback. Section IV then presents the proposed SFG approach and Section V demonstrates its application to illustrative examples, drawing conclusions in Section VI.

II. STABILITY OBJECTIVE

Negative-feedback loops in switching supplies regulate their outputs by sampling them and using that information to modulate, to whatever extent is necessary, the duty cycle \( d_L \) of the main switch in the power stage. Determining how a small-signal perturbation propagates through the loop to affect \( d_L \) and the output being regulated therefore examines the dynamics of the system, in other words, its ac response [1]. Ensuring the system is stable amounts to preventing signals from phase-shifting (around the loop) to 180° at unity-gain frequency \( f_{g_{in}} \), which in common terms translates to including phase margin in the response [14]. As a result, understanding what poles and zeros the switch-mode power stage introduces is crucial in establishing stable operating conditions.

This research is supported by Linear Technology Corporation, Milpitas, CA.

III. CONVENTIONAL POWER-STAGE AC MODELS

A. State-Space Averaging (SSA)

Switching converters are comprised of linear sub-circuits designed to operate in switch-mode for the purpose of reducing power losses. In state-space averaging (SSA), state-space-matrix differential equations from each linear sub-circuit are averaged across a switching cycle \( T_{SW} \) to eventually describe the small-signal behavior of the system [2-6]. Averaging the response in this manner filters transient effects so the model becomes invalid near the switching frequency \( f_{SW} \). Nevertheless, because developing state-space equations for a linear response is always possible, SSA is general, systematic, and comprehensive. The main challenges here are (i) the frequency-band limitation mentioned, (ii) extracting the ac response of the switching converters from SSA equations is involved, and (iii) although the equations describe the dynamics of the system, they do not often provide physical insight into the origins of the poles and zeros that result.

B. Circuit Averaging

Circuit-averaged models derive linear-equivalent circuits from averaged input-output signals across a nonlinear block, like the switches and diodes of a dc–dc converter [7-8]. In other words, they extract an electrical model comprised of ideal transformers and linearly dependent voltage and current sources from what amounts to SSA-inspired equations. Although this approach presents a circuit, which is operationally more intuitive than SSA equations (with respect to the derived model), it still suffers from the deficiencies attached to SSA because it relies on similarly derived equations.

C. Flow Graph

Flow graphs translate SSA equations into graphical form with nodes and connecting branches [9-13]. Each branch in the graph represents a small-signal transfer function that, when combined with the remaining branches, collectively describe the overall response of the entire loop. Although ascertaining the ac cause-effect relationship between adjacent signals in a flow graph is visually more straightforward than in the previous two approaches, the technique still relies on SSA equations, which means it ultimately suffers from the same shortcomings. Additionally, flow graphs in the literature are invalid in DCM, and developing DCM models from conventional equation-based means is complicated.

IV. PROPOSED OPERATION-BASED SIGNAL-FLOW GRAPH

A. Approach

The proposed signal-flow graph (SFG) recognizes the ease with which flow graphs in [9-13] convey information so it adopts a similar display format but derives its “branching” relationships directly from the operation of the circuit, ascribing more insight into the graph. The idea is for SFGs to literally trace ac-signal paths across a switching converter circuit and replace its constituent circuit blocks (e.g., switched inductors, capacitors, etc.) with their corresponding branching equivalents. Since the approach extracts the SFG from circuit-produced waveforms, not from the complex matrix of state-space differential equations, the proposed method is able to develop a unified SFG that applies to both CCM and DCM operation, allowing designers to more easily grasp and design for the stability requirements of switching converters across operation modes. As in [1-13], though, the proposed approach still “averages” the response across a switching cycle so its results remain valid for frequencies that fall below \( f_{SW} \).

In the paper, (i) lower case variables with upper case subscripts (e.g., \( i_L \)) include dc and ac components, (ii) all upper case variables (e.g., \( I_L \))...
describe dc signals only, (iii) all lower case variables (e.g., \( i_z \)) represent ac signals only, and (iv) lower case variables that include “(avg)” refer to per-cycle averages of ac signals.

### B. SFGs for Impedances

The foregoing SFG traces how small-signal variations propagate across a circuit and shows how circuit elements affect them. An impedance Z, for example, converts the voltage across its terminals \( v_Z \) into a current \( i_Z \) that is equal to \( v_Z/Z \) so its SFG representation (Fig. 1a) depicts ac signals \( v_Z \) and \( i_Z \) and connects them with a \( 1/Z \) branch. Conversely, Z transforms \( i_Z \) to \( v_Z \) with a Z branch (i.e., \( v_Z = Z i_Z \)) (Fig. 1a). This means resistors (R), inductors (L), and capacitors (C) similarly relate currents and voltages with \( R, sL, \) and \( 1/sC \) branches, respectively (Fig. 1b-d).

![Fig. 1. SFG equivalents for impedances and related passives.](image)

### C. Physical Interpretation of Poles and Zeros

Since the purpose of the graph is to illustrate how a circuit affects the ac-signal response, understanding how poles and zeros arise is important. From control theory, the effect of a pole or a zero is to attenuate or intensify signal strength at 20dB per decade of frequency. Said differently, poles draw energy away from a passing signal while zeros contribute energy [14]. Consider, for example, how shunting capacitor \( C_0 \) in Fig. 2a shunts the current (energy) away from resistor \( R_0 \) to ground as the impedance across \( C_0 \) decreases with increasing frequencies. The net effect is for \( v_o \) to decrease with frequency at –20dB per decade, which amounts to the effect of a pole.

![Fig. 2. Shunting capacitors produce poles and zeros.](image)

Similarly, \( C_0 \) in Fig. 2b also draws current (energy) away from \( R_o \), producing the effect of a pole. The difference here is the series resistor \( R_s \) ultimately limits the current to \( v_o/R_s \) when \( C_0 \) becomes, for all practical purposes, an ac short. In other words, \( C_0 \) no longer has a shunting effect on \( v_o \) when \( 1/sC_0 \) drops below \( R_s \), which equates to saying \( R_s \) and \( C_0 \) cancel the pole \( R_s C_0 \) produced in the first place. Control theory dictates a zero cancels a pole so series resistor \( R_s \) in Fig. 2b produces a zero.

More generally, zeros feed additional energy to a passing signal, or in the case of Fig. 2b, stop a pole from subtracting energy. From a circuit perspective, paths that feed-forward current (e.g., \( i_{FF} \)) around an existing (main) path and its related current (e.g., \( i_O' \)), as feed-forward transconductors \( G_{FF} \)’s exemplify in Fig. 3, can add sufficient energy to \( R_O \) to increase \( v_O \), producing the effect of a zero (when \( i_{FF} \) exceeds \( i_O' \)). Note, however, Fig. 3b’s \( -G_{FF} \) inverts the signal with respect to the main path (i.e., \( i_{FF} \) is out of phase with \( i_O' \)), which is detrimental when considering negative feedback relies on no inversion (+\( G_{FF} \)) across the block. The out-of-phase, feed-forward path in Fig. 3b therefore contributes a signal-strength intensifying zero that subtracts phase, which control theory calls a right-half-plane (RHP) zero. On the other hand, the in-phase, feed-forward path in Fig. 3a introduces a more benign left-half-plane (LHP) zero.

![Fig. 3. Feed-forward paths produce (a) LHP and (b) RHP zeros.](image)

### D. SFG for a Switched Inductor

**CCM and DCM Operation:** Switched inductors in dc-dc converters transfer energy from a source to one or several outputs in alternating energizing and de-energizing cycles. Inductor current \( i_L \) (Fig. 4), as a result, increases when applying a positive energizing voltage \( V_{EN} \) across \( L_o \) at a rate of \( V_{EN}/L_o \) and decreases when applying a negative de-energizing voltage \( V_{DE} \) at \( V_{DE}/L_o \). CCM refers to \( L_o \) conducting current continuously across its switching period \( T_{SW} \) (Fig. 4a), which means \( T_{SW} \) is literally \( L_o \)'s conduction time \( t_c \). DCM, in contrast, implies \( L_o \) ceases to conduct current before \( T_{SW} \) ends (Fig. 4b), which corresponds to \( L_o \) fully exhausting its stored energy before reaching the end of \( L_o \)'s switching cycle. In other words, \( t_c \) is a fraction of \( T_{SW} \).

![Fig. 4. (a) CCM and (b) DCM waveforms of a switched inductor \( L_o \).](image)

**Equivalent SFG:** Since \( t_c \) in CCM is \( T_{SW} \), small signal \( t_c \) is zero and small positive increments in inductor duty cycle \( d_l \) increase and decrease energizing and de-energizing times, respectively, by \( d_l T_{SW} \). As a result, there is remnant energy in \( L_o \) at the end of \( T_{SW} \) that increases average (per cycle) inductor voltage \( V_{L(avg)} \):

\[
y_{L(avg)} = d_l (V_{EN} - V_{DE})
\]

Therefore, inductor current \( i_{L(avg)} \) to subsequently increase:

\[
i_{L(avg)} = \frac{V_{L(avg)}}{L_o}
\]

Hence, in CCM, as shown in Fig. 5a, \( d_l \) controls \( i_{L(avg)} \) by modulating the voltage \( V_{L(avg)} \). Since \( i_{L(avg)} \) decreases linearly with frequency because of inductor impedance \( sL_o \), this signal path introduces a pole.

![Fig. 5. (a) Partial SFG for a switched inductor \( L_o \) and (b) its equivalent duty-cycle controlled current source representation.](image)

On the other hand, because \( t_c \) in DCM is a fraction of \( T_{SW} \), both energizing and de-energizing times increase (by \( d_l T_{SW} \) and \( t_c \)) with an incremental increase in \( d_l \) (Fig. 4b). Fully depleting \( L_o \)'s stored energy at the end of \( t_c \) means \( V_{L(avg)} \) remains unchanged (at zero). Conduction time \( t_c \), however, is no longer constant (at \( T_{SW} \)) and changes with \( d_l \):

\[
t_c = \frac{d_l}{T_{SW}} \left( \frac{T_{SW}}{t_c} \right)
\]

setting how much charge (q) \( L_o \) transfers per \( T_{SW} \) (i.e., \( i_{L(avg)} \)):

\[
i_{L(avg)} = \frac{d_l}{T_{SW}} \left( \frac{V_{L(avg)}}{L_o} \right)
\]

where \( i_{L(avg)} \) is \( i_L \)'s peak ripple current (from Fig. 4b). Therefore, in DCM (Fig. 5a), \( d_l \) controls \( i_{L(avg)} \) by modulating \( t_c \). Notice \( L_o \) in DCM no longer generates a pole, as conventional analyses corroborate [3-7]. Also note that \( V_{L(avg)} \) being zero in DCM nulls the effects of the CCM path, just as \( t_c \) being zero in CCM nulls those of the DCM counterpart. Although the transfer function differs, \( L_o \) is ultimately a duty-cycle controlled current source representation.

![Impedance \( Z_{Lout} \) reduces to \( sL_o \) in CCM because \( L_o \) always conducts current, which means \( L_o \)'s complete SFG equivalent in CCM (Fig. 6) not only connects \( d_l \) to \( V_{L(avg)} \) with \( V_{EN} \) and \( V_{DE} \) to \( i_{L(avg)} \) with 1/sL_o branches but also \( V_{L(avg)} \) to \( i_{L(avg)} \) via a \(-1/sL_o \) branch.](image)
However, impedance $Z_{\text{out}}$ is no longer $sL_o$ in DCM, because $i_L$ is discontinuous. Applying a test voltage $V_{\text{avg}}$ to the output as in Fig. 7, and deriving the resulting charge-per-cycle current $i_{\text{avg}}$ reveals

$$Z_{\text{out}} = \frac{V_{\text{avg}}}{i_{\text{avg}}}, \quad \frac{V_{\text{avg}}}{i_{\text{avg}}} = \frac{2T_c}{T_{\text{SW}}}, \quad \frac{V_{\text{avg}}}{i_{\text{avg}}} \frac{T_{\text{SW}}}{T_c} \quad (6)$$

where $T_{\text{LO}}$, a fraction of $T_{\text{SW}}$, is the amount of the time that $L_o$ conducts because of the output voltage of $L_O$. Hence, $L_o$'s SFG not only connects $d_i$ and $v_{\text{LO}}$ with $T_{\text{C}}/D_{\text{L}}$ and $t_c$ to $i_{\text{avg}}$ with $-T_{\text{C}}/2L_{\text{DC}}$ branches in DCM but also $V_{\text{avg}}$ to $i_{\text{avg}}$ with $-1/sL_{\text{O}}$ branch (Fig. 6).

In DCM, however, when the duty cycle is increased by $d_i$, all the increased charge (energy) in $L_o$ ($q_i$ in Fig. 4b) flows to the output ($q_o$ in Fig. 8b), as shown in Fig. 8b, so $i_{\text{avg}}$ is $i_{\text{avg}}$. The SFG equivalent for $D_0$ (shown in Fig. 8c) therefore relates $i_{\text{avg}}$ and $i_{\text{avg}}$ with $T_o/T_c$ and 1 branches in CCM and DCM, respectively.

As shown in Fig. 8a-b, an incremental increase in duty cycle $d_i$ not only (i) increases the energizing time by $d_i T_c$, and increases, as a result, $i_{\text{avg}}$ and $i_{\text{avg}}$, but also (ii) decreases $T_{\text{C}}$ by the same factor, effectively decreasing $i_{\text{avg}}$ in consequence. Because $D_0$'s feedforward impact on $i_{\text{avg}}$ produces an opposing effect with respect to $i_{\text{avg}}$'s $d_i$ (i.e., this out-of-phase feedforward path produces a RHP zero ($z_{\text{RHP}}$)). In this case, out-of-phase feedforward current $i_{\text{avg}}$ is the charge not transferred ($q_i$ in Fig. 8b) over $T_{\text{SW}}$, relating $d_i$ and $i_{\text{avg}}$ via $i_{\text{avg}}$ in the SFG with a $-1/sL_{\text{C}}$ branch.

$$i_{\text{avg}} = -d_i \frac{T_{\text{C}}}{T_{\text{SW}}} \quad \frac{i_{\text{avg}}}{T_{\text{C}}} \quad (8)$$

The resulting $z_{\text{RHP}}$ occurs at the frequency when $i_{\text{avg}}$ begins to exceed $i_{\text{avg}}$'s portion of $i_{\text{avg}}$ ($i_T$ in Fig. 3b or $i_{\text{avg}}$). Since $T_c$ equals $T_{\text{SW}}$ in CCM, $T_c/T_{\text{SW}}$ reduces to 1, $i_{\text{avg}}$ to $i_{\text{avg}}$, and $z_{\text{RHP}}$ to the constant contained below:

$$i_{\text{avg}} = \frac{V_{\text{SW}}}{sL_{\text{O}}}, \quad \frac{V_{\text{SW}}}{sL_{\text{O}}} = \frac{V_{\text{SW}}}{sL_{\text{O}}} \quad \left(\frac{V_{\text{SW}}}{sL_{\text{O}}} \right) \quad (9)$$

Note the loading effect of the switched inductor is neglected here because $Z_{\text{out}}$ near $z_{\text{RHP}}$ (i.e., at high frequencies) is typically much greater than the loading impedance.

In DCM, as shown in the SFG illustrated in Fig. 8c, $i_{\text{avg}}$ is higher to the point it always exceeds $i_{\text{avg}}$.

$$i_{\text{avg}} > i_{\text{avg}} \quad \left(\frac{i_{\text{avg}}}{D_i} \right) \quad (10)$$

Since $i_{\text{avg}}$ cannot invert $i_{\text{avg}}$, the RHP zero present in CCM disappears in DCM, as corroborated by [3-7]. In the end, $i_{\text{avg}}$ flows to $v_o$ so output capacitor $C_0$ and loading output resistor $R_0$ ultimately convert $i_{\text{avg}}$ to $V_o$ with a $(1/sC_0)R_0$ branch.

**V. VALIDATION**

A. Buck Converter

The switched inductance $L_o$ in a buck converter (Fig. 9a) transfers energy from input $V_{\text{IN}}$ to output $V_o$. Operationally, $L_o$ energizes from $V_{\text{IN}}$ to $V_o$ and de-energizes from $V_o$ to ground so $V_{\text{EN}} = V_{\text{IN}} - V_o$ and $V_{\text{DE}} = 0$-$V_o$, as shown in Fig. 9b's SFG when relating $d_i$ and $i_{\text{avg}}$. Since $L_o$ is directly connected to $V_o$, output conduction time $t_o$ equals $L_o$'s conduction time $t_c$ and $i_{\text{avg}}$ is $i_{\text{avg}}$. As a result, $V_{\text{avg}}$ and $i_{\text{avg}}$ connect via a $T_{\text{C}}/T_{\text{SW}}$ branch, $V_{\text{avg}}$ and $i_{\text{avg}}$ with $-1/sL_{\text{O}}$ and $-1/sL_{\text{O}}$ branches for CCM and DCM, respectively, and $i_{\text{avg}}$ and $i_{\text{avg}}$ with a $1/sC_0|R_0$ branch, as shown collectively in Fig. 9b.

Before deriving $V_{\text{avg}}/d_i$ to ascertain the stability of the system, note a negative-feedback control loop (that is not shown in Fig. 9a) senses and amplifies $V_{\text{avg}}$ to set $d_i$. As a result, at high frequency, $L_o$'s input ($d_i$) will dominantly set $i_{\text{avg}}$ and the SFG path carrying $V_{\text{avg}}$ signal can be neglected. Hence, in CCM, $V_{\text{avg}}/d_i$ can be simplified to the product of the CCM branches that connect $V_{\text{avg}}$ and $d_i$ in Fig. 9b's SFG:

$$\frac{V_{\text{avg}}}{d_i} = \frac{V_{\text{IN}}}{sL_{\text{O}}}, \quad \left(\frac{V_{\text{IN}}}{sL_{\text{O}}} \right) \quad (11)$$

which means the buck converter's power stage introduces a complex-conjugate pair of poles approximately at $1/2(\text{LO})^{1/2}$. In DCM, on the other hand, $V_{\text{avg}}/d_i$ reduces to

$$\frac{V_{\text{avg}}}{d_i} = \frac{T_c}{D_i}, \quad \left(\frac{V_{\text{avg}}}{d_i} \right) \quad (12)$$

because $L_{\text{C}}/2$ is the charge flowed to the output during $T_{\text{SW}}$, and

$$\frac{1}{2} = \frac{V_o}{R} \quad \left(\frac{1}{2} \right) \quad (13)$$

the power stage only introduces one pole approximately at $1/2\pi R_0$. Had the path carrying $V_{\text{avg}}$ not been neglected, the SFG would have produced the following more accurate results:

$$\frac{V_{\text{avg}}}{d_i} = \frac{V_{\text{IN}}}{1 + sL_{\text{O}}}, \quad \left(\frac{V_{\text{IN}}}{1 + sL_{\text{O}}} \right) \quad (14)$$

and

$$\frac{V_{\text{avg}}}{d_i} = \frac{2V_o}{D_i}, \quad \left(\frac{2V_o}{D_i} \right) \quad (15)$$

which corroborate the SSA relationships derived in [6]. The SPICE-generated frequency-response plots shown in Fig. 10a-b further illustrate and corroborate these results.
In DCM, not only does 1/sLO disappear from dIL to io(avg) but io(avg) is also large enough to never allow its feed-forward counterpart to overwhelm it, which means the RHP zero present in CCM is absent in DCM. The ac response therefore reduces to

$$\frac{v_{o(avg)}}{d} = \frac{v_o}{D_i} \left(1 + \frac{1}{s\left(\frac{R_R C_O}{2}\right)}\right)$$

which only has one pole at 1/sR_R C_O. These results correspond to those derived from SSA equations in [6] and emulate the SPICE results shown in Fig. 12a-b.

VI. CONCLUSIONS

The proposed operation-based signal-flow graph (SFG) not only easily conveys ac information graphically but also derives its connecting relationships directly from the operation of the circuit, from its defining waveforms, ascribing more insightful information into the graph and applying to both continuous- and discontinuous-conduction modes (CCM and DCM). Extracting the loop gain and presence of poles and zeros in complicated switch-mode circuits in this way allows the designer to more easily examine and optimize the frequency response of otherwise considerably complicated and abstract state-space-averaged equations. The ultimate and perhaps more important benefit, however, is the ability to design a higher performance (e.g., higher bandwidth, higher accuracy, etc.) switching supply, which is increasingly critical and difficult in state-of-the-art applications like wireless portable devices and energy harvesting ICs.

REFERENCES


